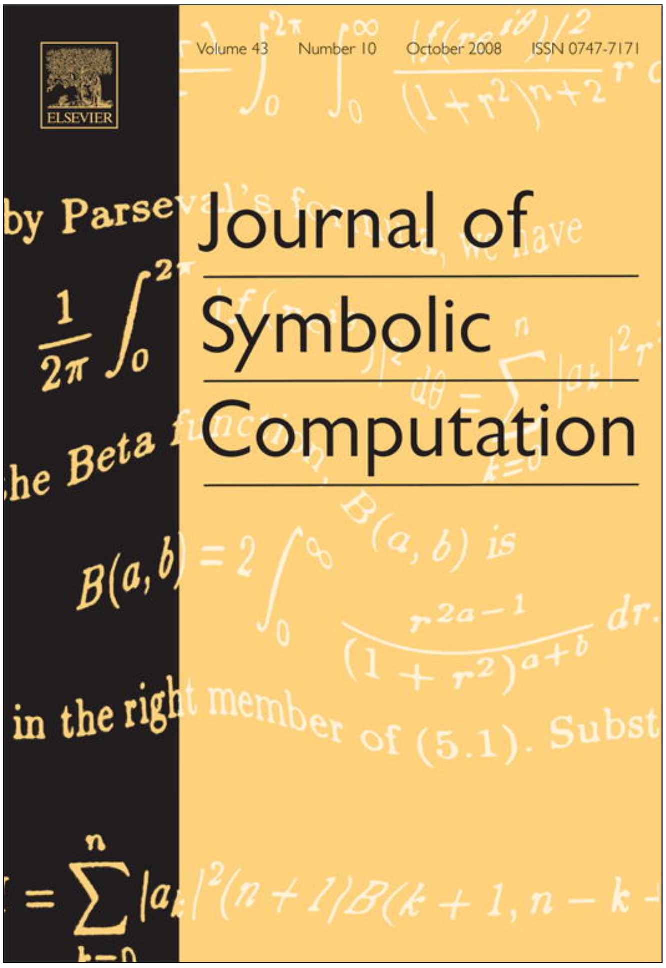
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Computing difference-differential dimension polynomials by relative Gröbner bases in difference-differential modules[☆]

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Abstract

In this paper we present a new algorithmic approach for computing the Hilbert function of a finitely generated difference-differential module equipped with the natural double filtration. The approach is based on a method of special Gröbner bases with respect to "generalized term orders" on $\mathbb{N}^m \times \mathbb{Z}^n$ and on difference-differential modules. We define a special type of reduction for two generalized term orders in a free left module over a ring of difference-differential operators. Then the concept of relative Gröbner bases w.r.t. two generalized term orders is defined. An algorithm for constructing these relative Gröbner bases is presented and verified. Using relative Gröbner bases, we are able to compute difference-differential dimension polynomials in two variables.

Keywords: Relative Gröbner basis; Generalized term order; Difference-differential module; Difference-differential dimension polynomial

1. Introduction

The notion of Gröbner basis, being a powerful tool to solve various problems by algorithmic way in polynomial ideal theory, has been explored in differential algebra and difference-

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differential algebra by many researchers. Although the attempt to imitate Gröbner bases in the context of differential ideals of a ring of differential polynomials has been unsuccessful to date, the theory of Gröbner bases in free modules over various rings of differential operators has been developed, see Noumi (1988), Takayama (1989), Oaku and Shimoyama (1994), Carra Ferro (1997), Insa and Pauer (1998), Pauer and Unterkircher (1999), Levin (2000) and Zhou and Winkler (2006). It has been shown that the notion of Gröbner basis is essential for many problems of linear difference-differential equations such as the dimension of the space of solutions and the computation of difference-differential dimension polynomials.

The concept of the differential dimension polynomial was introduced in Kolchin (1964) as a dimensional description of some differential field extension. Johnson (1974) proved that the differential dimension polynomial of a differential field extension coincides with the Hilbert polynomial of some filtered differential module. This result allowed to compute differential dimension polynomials using the Gröbner basis technique. Since then various problems of differential algebra involving differential dimension polynomials have been studied; see Levin and Mikhalev (1987) and Kondrateva et al. (1998). The concepts of the difference dimension polynomial and the difference-differential dimension polynomial were introduced first in Levin (1978) and Dzhavadov (1979). Some additional properties of such a polynomial can be found in Chapters 6 and 8 of Kondrateva et al. (1998). These polynomials play the same role in difference algebra (resp. difference-differential algebra) as Hilbert polynomials in commutative algebra or differential dimension polynomials in differential algebra. The notion of difference-differential dimension polynomial can be used for the study of dimension theory of difference-differential field extensions and of systems of algebraic difference-differential equations.

By the classical Gröbner basis method for computing Hilbert polynomials, one can study difference-differential dimension polynomials $\phi(t)$ associated with a difference-differential module M. This approach is based on the fact that the ring of difference-differential operators over the difference-differential field R is isomorphic to the factor ring of the ring of non-commutative polynomials $R[x_1, \dots, x_{m+2n}]$ modulo the ideal I, where $x_i = \delta_i$, $x_i a = a x_i + \delta_i(a)$ $x_{m+j} = \alpha_j$ (an isomorphism on R), $x_{m+j} a = \alpha_j(a) x_{m+j}$, $x_{m+n+j} = \alpha_j^{-1}$, $x_{m+n+j} a = \alpha_j^{-1}(a) x_{m+n+j}$ for $1 \le i \le m$, $1 \le j \le n$, and $a \in R$, and I is generated by the polynomials $x_{m+j} x_{m+n+j} - 1$ for $1 \le j \le n$. However, a similar approach to difference-differential dimension polynomials in two variables is unsuccessful. Levin (2000) investigated the difference-differential dimension polynomials in two variables by the characteristic set approach. Levin also gave an algorithm to compute the dimension polynomials if the characteristic sets have been obtained. The method of Levin is rather delicate but no general algorithm for computing the characteristic set is given. In his recent paper Levin (2007) deals with difference-differential operators, but does not directly consider their inverses. In this paper we explicitly consider the inverses of difference operators (automorphisms), and therefore we have to generalize term orders and to include also terms with negative exponents. The concept of Gröbner bases w.r.t. several orderings in Levin (2007) is rather involved. We present an alternative concept of relative Gröbner bases. Based on this simpler concept we can also exhibit examples of the theory.

In this paper we introduce a new concept, relative difference-differential Gröbner bases, for algorithmically computing the difference-differential dimension polynomials in two variables. Our notion of relative Gröbner basis is based on two generalized term orders on $\mathbb{N}^m \times \mathbb{Z}^n$. We define a special type of reduction for two generalized term orders in a free left module over a ring of difference-differential operators. Then the concept of relative Gröbner bases w.r.t. two generalized term orders is defined. An algorithm for constructing these Gröbner basis counterparts is presented and verified. Using the relative Gröbner basis algorithm, it is possible to compute difference-differential dimension polynomials in two variables. The results obtained improve essentially theories of Levin (2000), in which the existence of the difference-differential dimension polynomial was proved via characteristic set.

This paper is divided into 4 sections. Section 1 is introduction and Section 2 is preliminaries. Most material in the preliminaries is based on Levin (2000) and Zhou and Winkler (2006). In Section 3 we design the relative reduction algorithm, give the definition of relative Gröbner bases and S-polynomials, as well as the Buchberger algorithm for computation of relative Gröbner bases. Some results are proved already in Zhou and Winkler (2006). In Section 4 we describe the approach to compute difference-differential dimension polynomials in two variables via relative Gröbner bases. We also need Levin's theorem on the existence of the difference-differential dimension polynomial and the algorithm for counting suitable sets of "non-leading terms" $Card\ U_{r,s}$ (see Theorem 4.1).

2. Preliminaries

In this paper $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_-$ and $\mathbb{Q}$ will denote the sets of all integers, all nonnegative integers, all nonpositive integers, and all rational numbers, respectively. By a ring we always mean an associative ring with a unit. By the module over a ring A we mean a unitary left A-module.

Definition 2.1. Let R be a commutative noetherian ring, $\Delta = \{\delta_1, \ldots, \delta_m\}$ a set of derivations and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ a set of automorphisms of the ring R, which commute with each other; i.e. $\alpha \circ \beta = \beta \circ \alpha$ for all $\alpha, \beta \in \Delta \cup \Sigma$. Then R is called a *difference-differential ring* with the basic set of derivations Δ and the basic set of automorphisms Σ, or shortly a *Δ–Σ-ring*. If R is a field, then it is called a *Δ–Σ-field*. □

Throughout the paper we suppose that R is a Δ–Σ-field and elements of $\Delta \cup \Sigma$ are free generators of a commutative semigroup. Then Λ will denote the commutative semigroup of terms, i.e. elements of the form

$$\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n}, \tag{2.1}$$

where $(k_1, \ldots, k_m) \in \mathbb{N}^m$ and $(l_1, \ldots, l_n) \in \mathbb{Z}^n$. This semigroup contains the free commutative semigroup Θ generated by the set Δ and free commutative semigroup Γ generated by the set Σ.

Definition 2.2. Let R and Λ be as above. The free R-module generated by Λ is denoted by D. Elements of D are of the form

$$\sum_{\lambda \in \Lambda} a_\lambda \lambda, \tag{2.2}$$

where $a_\lambda \in R$ for all $\lambda \in \Lambda$ and only finitely many coefficients a_λ are different from zero. They will be called difference-differential operators (or shortly a Δ–Σ-operators) over R. Two Δ–Σ-operators $\sum_{\lambda \in \Lambda} a_\lambda \lambda$ and $\sum_{\lambda \in \Lambda} b_\lambda \lambda$ are equal if and only if $a_\lambda = b_\lambda$ for all $\lambda \in \Lambda$. □

The free R-module D can be equipped with a natural ring structure. It is called the ring of difference-differential operators (or shortly the ring of Δ–Σ-operators) over R. Note that

$$\delta a = a\delta + \delta(a), \quad \tau a = \tau(a)\tau, \tag{2.3}$$

for all $a \in R$, $\delta \in \Delta$, $\tau \in \Sigma \cup \{\sigma^{-1} | \sigma \in \Sigma\}$. The terms $\lambda \in \Lambda$ do not commute with the coefficients $a_\lambda \in R$.

A left D-module M is called a difference-differential module (or a Δ–Σ-module). If M is finitely generated as a left D-module, then M is called a finitely generated Δ–Σ-module.

Definition 2.3. A family of subsets $\{\mathbb{Z}_j^{(n)}, j = 1, \ldots, k\}$ of $\mathbb{Z}^n$ is called an *orthant decomposition* of $\mathbb{Z}^n$ and $\mathbb{Z}_j^{(n)}$ is called the *jth orthant* of the decomposition, if

$$\mathbb{Z}^n = \bigcup_{j=1}^{k} \mathbb{Z}_j^{(n)}$$

and for all $j = 1, \ldots, k$ the following conditions hold:

(i) $(0, \ldots, 0) \in \mathbb{Z}_j^{(n)}$, and $\mathbb{Z}_j^{(n)}$ does not contain any pair of inverse elements $c = (c_1, \ldots, c_n) \neq 0$ and $c^{-1} = (-c_1, \ldots, -c_n)$;
(ii) $\mathbb{Z}_j^{(n)}$ is a finitely generated subsemigroup of $\mathbb{Z}^n$, which is isomorphic to $\mathbb{N}^n$ as a semigroup;
(iii) the group generated by $\mathbb{Z}_j^{(n)}$ is $\mathbb{Z}^n$.

We extend orthant decompositions from $\mathbb{Z}^n$ to $\mathbb{N}^m \times \mathbb{Z}^n$:

Let $\{\mathbb{Z}_j^{(n)}, j = 1, \ldots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$. Then we call $\{\mathbb{N}^m \times \mathbb{Z}_j^{(n)}, j = 1, \ldots, k\}$ an *orthant decomposition* of $\mathbb{N}^m \times \mathbb{Z}^n$. □

Example 2.1. Let $\{\mathbb{Z}_1^{(n)}, \ldots, \mathbb{Z}_{2^n}^{(n)}\}$ be all distinct Cartesian products of n sets each of which is either $\mathbb{N}$ or $\mathbb{Z}_-$. This is an orthant decomposition of $\mathbb{Z}^n$. The set of generators of $\mathbb{Z}_j^{(n)}$ as a semigroup is

$$\{(c_1, 0, \ldots, 0), (0, c_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, c_n)\},$$

where c_j is either 1 or -1, $j = 1, \ldots, n$. We call this decomposition the *canonical orthant decomposition* of $\mathbb{Z}^n$. □

Definition 2.4. Let $\{\mathbb{Z}_j^{(n)}, j = 1, \ldots, k\}$ be an orthant decomposition of $\mathbb{Z}^n$. Let $E = \{e_1, \ldots, e_q\}$ be a set of q distinct elements. A total order $\prec$ on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is called a *generalized term order* on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ with respect to the decomposition, if the following conditions hold:

(i) $(0, \ldots, 0, e_i)$ is the smallest element in $\mathbb{N}^m \times \mathbb{Z}^n \times \{e_i\}$, $e_i \in E$,
(ii) if $(a, e_i) \prec (b, e_j)$, then for any c such that c and b are in the same orthant, $(a + c, e_i) \prec (b + c, e_j)$, where $a, b, c \in \mathbb{N}^m \times \mathbb{Z}^n$, $e_i, e_j \in E$. □

Example 2.2. Given the canonical orthant decomposition of $\mathbb{Z}^n$, an order "$\prec'$" in $E = \{e_1, \ldots, e_q\}$, for two elements $(a, e_i) = (k_1, \ldots, k_m, l_1, \ldots, l_n, e_i)$ and $(b, e_j) = (r_1, \ldots, r_m, s_1, \ldots, s_n, e_j)$ of $\mathbb{N}^m \times \mathbb{Z}^n \times E$ define:

$$|a|_1 = \sum_{j=1}^{m} k_j, \qquad |a|_2 = \sum_{j=1}^{n} |l_j|.$$

$$(a, e_i) \prec (b, e_j) \Longleftrightarrow (|a|_1, |a|_2, e_i, k_1, \ldots, k_m, |l_1|, \ldots, |l_n|, l_1, \ldots, l_n)$$

$$< (|b|_1, |b|_2, e_j, r_1, \ldots, r_m, |s_1|, \ldots, |s_n|, s_1, \ldots, s_n) \text{ in lexicographic order.}$$

Then "$\prec$" is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$. □

Let Λ be the semigroup of terms of the form (2.1). Since Λ is isomorphic to $\mathbb{N}^m \times \mathbb{Z}^n$ as a semigroup, a generalized term order "$\prec$" on $\mathbb{N}^m \times \mathbb{Z}^n$ induces an order on Λ. We call this a generalized term order on Λ. The notion of generalized term orders can be easily extended to finitely generated free D-modules. The following result can be found in Zhou and Winkler (2006).

Lemma 2.1. *Given an orthant decomposition of $\mathbb{Z}^n$ and a generalized term order "$\prec$" on $\mathbb{N}^m \times \mathbb{Z}^n \times E$, every strictly descending sequence in $\mathbb{N}^m \times \mathbb{Z}^n \times E$ is finite. In particular, any subset of $\mathbb{N}^m \times \mathbb{Z}^n \times E$ contains a smallest element.* $\square$

3. Relative Gröbner bases in finitely generated difference-differential modules

Let R be a Δ–Σ-field and D be the ring of Δ–Σ-operators over R, and let F be a finitely generated free D-module (i.e. a finitely generated free difference-differential module) with a set of free generators $E = \{e_1, \dots, e_q\}$. Then F can be considered as an R-module generated by the set of all elements of the form λe_i ($i = 1, \dots, q$, where $\lambda \in \Lambda$). This set will be denoted by ΛE and its elements will be called *terms* of F. In particular the elements of Λ will be called terms of D. If "$\prec$" is a generalized term order on $\mathbb{N}^m \times \mathbb{Z}^n \times E$ then "$\prec$" induces a generalized term order on ΛE.

It is clear that every element $f \in F$ has a unique representation as a linear combination of terms:

$$f = a_1 \lambda_1 e_{j_1} + \cdots + a_d \lambda_d e_{j_d} \tag{3.1}$$

for some nonzero elements $a_i \in R$ ($i = 1, \dots, d$) and some distinct elements $\lambda_1 e_{j_1}, \dots, \lambda_d e_{j_d} \in \Lambda E$.

Let "$\prec$" be a generalized term order on ΛE, $f \in F$ be of the form (3.1). Then

$$lt(f) := \max_{\prec}\{\lambda_i e_{j_i} | i = 1, \dots, d\}$$

is called the *leading term* of f. If $\lambda_i e_{j_i} = lt(f)$, then $lc(f) = a_i$ is called the *leading coefficient* of f.

Now we are going to construct a special reduction algorithm in the difference-differential module F. In what follows we always assume that an orthant decomposition of $\mathbb{Z}^n$ is given as well as a generalized term order with respect to this decomposition. We need some lemmas which have been proved in Zhou and Winkler (2006) to describe the various properties in difference-differential modules.

Definition 3.1. Let λ be of the form (2.1). Then the subset Λ_j of Λ,

$$\Lambda_j = \{\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \mid (l_1, \dots, l_n) \in \mathbb{Z}_j^{(n)}\},$$

where $\mathbb{Z}_j^{(n)}$ is the jth orthant of the decomposition of $\mathbb{Z}^n$, is called jth *orthant* of Λ. Let F be a finitely generated free D-module and ΛE be the set of terms of F. Then

$$\Lambda_j E = \{\lambda e_i \mid \lambda \in \Lambda_j, e_i \in E\}$$

is called jth *orthant* of ΛE. $\square$

Obviously, from Definition 2.3, if "$\prec$" is a generalized term order on Λ and $\xi \prec \lambda$, then $\eta\xi \prec \eta\lambda$ holds for any η in the same orthant as λ.

Lemma 3.1. *Let $\lambda \in \Lambda$ and $a \in R$, "$\prec$" be a generalized term order on $\Lambda E \subseteq D$. Then*

$$\lambda a = a'\lambda + \xi,$$

where $a' = \alpha(a)$ for some $\alpha \in \Gamma$ (see (2.1)), and if $a \neq 0$ then $a' \neq 0$; $\xi \in D$ with $lt(\xi) \prec \lambda$ and all terms of ξ are in the same orthant as λ. $\square$

In general $lt(\lambda f) = \lambda lt(f)$ does not hold, a slightly weaker property is expressed in the following lemma.

Lemma 3.2. *Let F be a finitely generated free D-module and $f \in F \setminus \{0\}$. Then the following assertions hold:*

(i) *If $\lambda \in \Lambda$, then $lt(\lambda f) = \max_{\prec}\{\lambda u_i\}$ where u_i are terms of f, and $lt(\lambda f) = \lambda u$ for a unique term u of f.*
(ii) *If $lt(f) \in \Lambda_j E$ then $lt(\lambda f) = \lambda lt(f) \in \Lambda_j E$ holds for any $\lambda \in \Lambda_j$.* $\square$

Lemma 3.3. *Let F be a finitely generated free D-module and $f \in F \setminus \{0\}$. Then for each j there exists some $\lambda \in \Lambda$ and a term u_j of f such that*

$$lt(\lambda f) = \lambda u_j \in \Lambda_j E.$$

Furthermore, the term u_j of f is unique: if $lt(\lambda_1 f) = \lambda_1 u_{j_1} \in \Lambda_j E$ and $lt(\lambda_2 f) = \lambda_2 u_{j_2} \in \Lambda_j E$ then $u_{j_1} = u_{j_2}$. We will write $lt_j(f)$ for the term u_j. $\square$

If $h = \sum_{i \in I} b_i \lambda_i \in D$ and $f = \sum_{j \in J} c_j u_j \in F$, then $hf = \sum_{i \in I, j \in J} b_i c'_j \lambda_i u_j$ (as in Lemma 3.1). Since some of the terms $\lambda_i u_j$ may be equal and vanish in hf, it would be problematic if $lt(hf) \prec \lambda_i u_j$ might occur for some λ_i and u_j. The following lemma asserts that this undesirable situation cannot occur.

Lemma 3.4. *Let $f \in F \setminus \{0\}$, $h \in D \setminus \{0\}$. Then $lt(hf) = \max_{\prec}\{\lambda_i u_k\}$ where λ_i are terms of h and u_k are terms of f. Therefore $lt(hf) = \lambda u$ for a unique term λ of h and a unique term u of f.* $\square$

Let "$\prec$" be a generalized term order on ΛE. For our purpose we need to consider a special type of reduction relative to another generalized term order "$\prec'$" on ΛE. An algorithm for the reduction is described in the following theorem.

Theorem 3.1. *Let "$\prec$" and "$\prec'$" be two generalized term orders on ΛE. Let $g_1, \ldots, g_p \in F \setminus \{0\}$ and $f \in F$. Then*

$$f = h_1 g_1 + \cdots + h_p g_p + r \tag{3.2}$$

for some elements $h_1, \ldots, h_p \in D$ and $r \in F$ such that

(i) *$h_i = 0$ or $lt_{\prec}(h_i g_i) \preceq lt_{\prec}(f)$, $i = 1, \ldots, p$; (By Lemma 3.4 this means that $\lambda u \preceq lt_{\prec}(f)$ for all terms λ of h_i and all terms u of g_i.)*
(ii) *$r = 0$ or $lt_{\prec}(r) \preceq lt_{\prec}(f)$ such that*

$$lt_{\prec}(r) \notin \{lt_{\prec}(\lambda g_i) \mid lt_{\prec'}(\lambda g_i) \preceq' lt_{\prec'}(r), \lambda \in \Lambda, i = 1, \ldots, p\}.$$

Proof. The elements $h_1, \ldots, h_p \in D$ and $r \in F$ can be computed as follows:
First set $r = f$ and $h_i = 0$, $i = 1, \ldots, p$.

While $r \neq 0$ and $lt_{\prec}(r) = lt_{\prec}(\lambda_i g_i)$ such that $lt_{\prec'}(\lambda_i g_i) \preceq' lt_{\prec'}(r)$ for an element $\lambda_i \in \Lambda$, then

$$\lambda_i g_i = c_i lt_{\prec}(\lambda_i g_i) + \xi_i$$
$$c_i lt_{\prec}(\lambda_i g_i) = \lambda_i g_i - \xi_i,$$

where $c_i = lc_{\prec}(\lambda_i g_i)$ and $lt_{\prec}(\xi_i) \prec lt_{\prec}(\lambda_i g_i)$. Therefore

$$r = lc_{\prec}(r)lt_{\prec}(r) + \cdots = lc_{\prec}(r)lt_{\prec}(\lambda_i g_i) + \cdots = \frac{lc_{\prec}(r)}{c_i}(\lambda_i g_i - \xi_i) + \cdots,$$

where all terms in $\cdots \prec lt_{\prec}(r) = lt_{\prec}(\lambda_i g_i)$.

Put $b_i = \frac{lc_{\prec}(r)}{c_i}$ and $r_i = \frac{lc_{\prec}(r)}{c_i} \cdot (-\xi_i) + \cdots$. Then

$$r = b_i \lambda_i g_i + r_i. \tag{3.3}$$

Now we may replace r by r_i and h_i by $h_i + b_i \lambda_i$. Since in each step we have

$$lt_{\prec}(r_i) \prec lt_{\prec}(\lambda_i g_i) \preceq lt_{\prec}(r) \preceq lt_{\prec}(f),$$

by Lemma 2.1, the algorithm above terminates after finitely many steps. This completes the proof. □

Definition 3.2. Let "$\prec$" and "$\prec'$" be two generalized term orders on ΛE. Let $g_1, \ldots, g_p \in F\backslash\{0\}$ and $f \in F$. Suppose that the Eq. (3.2) holds and that the conditions (i), (ii) in Theorem 3.1 are satisfied. If $r \neq f$ we say that f *can be* $\prec$*-reduced to* r *modulo* $\{g_1, \ldots, g_p\}$ *relative to* $\prec'$. In case $r = f$ and $h_i = 0$, $i = 1, \ldots, p$, we say that f *is* $\prec$*-reduced modulo* $\{g_1, \ldots, g_p\}$ *relative to* $\prec'$. □

Unlike the difference-differential reduction for one generalized term order in Zhou and Winkler (2006), in every step of the relative reduction we reduce the term $lt_{\prec}(r) = lt_{\prec}(\lambda_i g_i)$ only if $lt_{\prec'}(\lambda_i g_i) \preceq' lt_{\prec'}(r)$. This is why we call the reduction "relative to $\prec'$".

Example 3.1. Let the sets Δ and Σ consist of a single δ and a single σ, and let D be the ring of Δ–Σ-operators over R. Choose the canonical orthant decomposition on $\mathbb{Z}$ as in Example 2.1 and define the generalized term orders $\prec$ and $\prec'$ on terms of D as follows:

$$\delta^k \sigma^l \prec \delta^r \sigma^s \iff (|l|, k, l) < (|s|, r, s) \text{ in lexicographic order},$$
$$\delta^k \sigma^l \prec' \delta^r \sigma^s \iff (k, |l|, l) < (r, |s|, s) \text{ in lexicographic order}.$$

Given $f = \delta^3\sigma - \sigma^{-1}$, $g = \delta^2 + \sigma$, then $lt_{\prec}(f) = \delta^3\sigma = lt_{\prec}(\delta^3 g) = lt_{\prec}(\delta^5 + \delta^3\sigma)$. But $lt_{\prec'}(\delta^3 g) = \delta^5 \succ' lt_{\prec'}(f) = \delta^3\sigma$. So f is not $\prec$-reduced modulo g in the usual meaning, but f is $\prec$-reduced modulo g relative to $\prec'$. □

Definition 3.3. Let W be a submodule of the finitely generated free D-module F, $\prec$ and $\prec'$ be two generalized term orders on ΛE, and $G = \{g_1, \ldots, g_p\}$ a subset of $W\backslash\{0\}$. Then G is called a $\prec$*-Gröbner basis of* W *relative to* $\prec'$ iff every $f \in W\backslash\{0\}$ can be $\prec$-reduced to 0 modulo G relative to $\prec'$. We will call it shortly a *relative Gröbner basis* of W if no confusion is possible.

Obviously, $G = \{g_1, \ldots, g_p\} \subset W\backslash\{0\}$ is a $\prec$-Gröbner basis of W relative to $\prec'$ if and only if for every $f \in W\backslash\{0\}$ we have

$$lt_{\prec}(f) \in \{lt_{\prec}(\lambda g_i) \mid lt_{\prec'}(\lambda g_i) \preceq' lt_{\prec'}(f), \lambda \in \Lambda, i = 1, \ldots, p\}. \quad \square$$

Remark. If we take the two generalized term orders $\prec$ and $\prec'$ on ΛE as $\prec = \prec'$, then relative reduction will be the usual reduction and a relative Gröbner basis will be a usual difference-differential Gröbner basis as introduced in Zhou and Winkler (2006). So the concept of relative Gröbner basis is a generalization of Gröbner bases in difference-differential modules.

Proposition 3.1. *Let $\prec$ and $\prec'$ be two generalized term orders on ΛE, $G \subset W \setminus \{0\}$ a $\prec$-Gröbner basis of W relative to $\prec'$, and $f \in F$. Then the following assertions hold:*

(i) *G is a Gröbner basis of W w.r.t. $\prec$ and $\prec'$. So G generates the D-module W.*
(ii) *$f \in W$ if and only if $f = 0$ or f can be $\prec$-reduced to 0 modulo G relative to $\prec'$.*
(iii) *$f \in W$ is $\prec$-reduced modulo G relative to $\prec'$ if and only if $f = 0$.*

Proof. (i) If f can be $\prec$-reduced to 0 modulo G relative to $\prec'$, then f can be reduced to 0 modulo G w.r.t. $\prec$ in the classical way. Therefore G is a Gröbner basis of W w.r.t. $\prec$.

In order to see that G is also a Gröbner basis of W w.r.t. $\prec'$, we consider an arbitrary $f \in W \setminus \{0\}$. We write

$$f = h_1 g_1 + \cdots + h_p g_p$$

by the algorithm described in Theorem 3.1. Note that in every step of the relative reduction algorithm we have

$$lt_{\prec'}(\lambda_i g_i) \preceq' lt_{\prec'}(r).$$

From (3.3) we see that $lt_{\prec'}(r_i) \preceq' lt_{\prec'}(r)$. So if $lt_{\prec'}(r) \preceq' lt_{\prec'}(f)$ then $lt_{\prec'}(r_i) \preceq' lt_{\prec'}(f)$. In the first step we set $r = f$. So in every step we have $lt_{\prec'}(r_i) \preceq' lt_{\prec'}(f)$.

Moreover, if $lt_{\prec'}(h_i g_i) \preceq' lt_{\prec'}(\lambda_i g_i)$ then $lt_{\prec'}((h_i + b_i \lambda_i) g_i) \preceq' lt_{\prec'}(\lambda_i g_i)$. This means that in every step we have $lt_{\prec'}(h_i g_i) \preceq' lt_{\prec'}(r_i) \preceq' lt_{\prec'}(f)$ since in the first step we set $h_i = 0$.

We conclude that f can be reduced to 0 modulo G w.r.t. $\prec'$, i.e. G is a Gröbner basis of W w.r.t. $\prec'$.

(ii) and (iii) are obvious from Theorem 3.1 and Definition 3.3. $\square$

Remark. Proposition 3.1(i) asserts that a relative Gröbner basis of W must be a Gröbner basis of W w.r.t. $\prec$ and $\prec'$. But the reverse conclusion is not true. For example, if $\{g_1, \ldots, g_p\}$ and $\{g'_1, \ldots, g'_q\}$ are Gröbner bases of W w.r.t. $\prec$ and $\prec'$ resp., then $\{g_1, \ldots, g_p, g'_1, \ldots, g'_q\}$ is a Gröbner basis of W w.r.t. $\prec$ and $\prec'$. But it need not be a relative Gröbner basis of W.

Example 3.2. Let $\prec$ and $\prec'$ be two generalized term orders on ΛE. If W is generated by one element $g \in F \setminus \{0\}$, then any finite subset G of $W \setminus \{0\}$ containing g is a relative Gröbner basis of W. In fact, $0 \neq f \in W$ implies $f = hg$ for some $0 \neq h \in D$. By Lemma 3.4, $lt_{\prec}(f) = \lambda u = \max_{\prec}\{\lambda_i u_k\}$ for a term λ of h and a term u of g, where λ_i are terms of h and u_k are terms of g. Then $lt_{\prec}(f) = lt_{\prec}(\lambda g)$. Similarly, we have also that $lt_{\prec'}(f) = \max_{\prec'}\{\lambda_i u_k\} \succeq' lt_{\prec'}(\lambda g)$. By Definition 3.3, G is a relative Gröbner basis of W. $\square$

In Zhou and Winkler (2006) we have presented an algorithm for computing Gröbner bases of difference-differential modules. In a similar manner we will now construct an algorithm for computing relative Gröbner bases.

Definition 3.4. Let F be a finitely generated free D-module and $f, g \in F \setminus \{0\}$. Let $\prec$ be a generalized term order on ΛE. For every Λ_j let $V(j, f, g)$ be a finite system of generators of the $R[\Lambda_j]$-module

$${}_{R[\Lambda_j]}\langle lt(\lambda f) \in \Lambda_j E \mid \lambda \in \Lambda\rangle \cap {}_{R[\Lambda_j]}\langle lt(\eta g) \in \Lambda_j E \mid \eta \in \Lambda\rangle.$$

Then for every generator $v \in V(j, f, g)$

$$S(j, f, g, v) = \frac{v}{lt_j(f)} \frac{f}{lc_j(f)} - \frac{v}{lt_j(g)} \frac{g}{lc_j(g)}$$

is called an *S-polynomial* of f and g with respect to j and v. $\square$

Theorem 3.2 (*Generalized Buchberger Theorem in Zhou and Winkler (2006)*). *Let F be a free D-module and $\prec$ be a generalized term order on ΛE, G be a finite subset of $F \backslash \{0\}$ and W be the submodule in F generated by G. Then G is a Gröbner basis of W if and only if for all Λ_j, for all $g_i,\ g_k \in G$ and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ can be reduced to 0 by G.* $\square$

On the basis of Theorem 3.2, we can construct the algorithm of computing relative Gröbner bases. Let F be a free D-module, $\prec$ and $\prec'$ be two generalized term orders on ΛE. We will denote the S-polynomials with respect to $\prec$ and $\prec'$ by $S(j, g_i, g_k, v)$ and $S'(j, g_i, g_k, v)$ respectively.

Theorem 3.3. *Let F be a free D-module, $\prec$ and $\prec'$ be two generalized term orders on ΛE, G be a finite subset of $F \backslash \{0\}$ and W be the submodule in F generated by G. Then G is a $\prec$-Gröbner basis of W relative to $\prec'$ if and only if G is a Gröbner basis with respect to $\prec'$ of W and for all Λ_j, for all $g_i,\ g_k \in G$ and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ with respect to $\prec$ can be $\prec$-reduced to 0 modulo G relative to $\prec'$.*

In other words, G is a $\prec$-Gröbner basis relative to $\prec'$ if and only if all $S'(j, g_i, g_k, v)$ can be reduced (w.r.t. $\prec'$) to 0 by G and all $S(j, g_i, g_k, v)$ can be $\prec$-reduced to 0 modulo G relative to $\prec'$.

Proof. Suppose that G is a $\prec$-Gröbner basis of W relative to $\prec'$. Since $S(j, g_i, g_k, v)$ is an element of W, then it follows from Proposition 3.1(ii) that $S(j, g_i, g_k, v)$ can be $\prec$-reduced to 0 modulo G relative to $\prec'$. Also, G is a Gröbner basis with respect to $\prec'$ of W by Proposition 3.1(i).

On the other hand, let G be a finite subset of $F \backslash \{0\}$ and W be the submodule in F generated by G. Suppose that for all Λ_j, for all $v \in V(j, g_i, g_k)$ and for all $g_i,\ g_k \in G$, the S-polynomials $S(j, g_i, g_k, v)$ can be $\prec$-reduced to 0 by G relative to $\prec'$, and G is a Gröbner basis with respect to $\prec'$ of W. For any $f \in W \backslash \{0\}$ we have to show that there are some $\lambda \in \Lambda$ and $g \in G$ such that $lt_{\prec}(f) = lt_{\prec}(\lambda g)$ and $lt_{\prec'}(\lambda g) \preceq' lt_{\prec'}(f)$.

Since W is generated by G and G is a Gröbner basis with respect to $\prec'$ of W, we have

$$f = \sum_{g \in G} h_g g$$

for some $\{h_g\}_{g \in G} \subseteq D$ such that

$$lt_{\prec'}(h_g g) \preceq' lt_{\prec'}(f).$$

In the following we denote $lt_{\prec}(f)$ shortly by $lt(f)$. Let $u = \max_{\prec}\{lt(h_g g) \mid g \in G\}$. We may choose the family $\{h_g \mid g \in G\}$ such that u is minimal, i.e. if $f = \sum_{g \in G} h'_g g$ then $u \preceq \max_{\prec}\{lt(h'_g g) \mid g \in G\}$. Note that $u \succeq lt(\lambda g)$ for all terms λ of h_g and all $g \in G$ by Lemma 3.4. Also we have $lt_{\prec'}(\lambda g) \preceq' lt_{\prec'}(f)$ for all terms λ of h_g and all $g \in G$.

If $lt(f) = u = lt(h_g g)$ for some $g \in G$, then it follows from Lemma 3.4 that there is a term λ of h_g such that $lt(f) = lt(\lambda g)$ and $lt_{\prec'}(\lambda g) \preceq' lt_{\prec'}(f)$. Therefore the proof would be completed. Hence it remains to show that $lt(f) \prec u$ cannot hold.

Suppose that $lt(f) \prec u$ and let $B = \{g \mid lt(h_g g) = u \succ lt(f)\}$. Then by Lemma 3.4 there is a unique term λ_g of h_g, $g \in B$, such that $u = lt(\lambda_g g) \succ lt(\eta_g g)$ for any terms $\eta_g \neq \lambda_g$ of h_g. Let c_g be the coefficient of h_g at λ_g. We have

$$f = \sum_{g\in B} h_g g + \sum_{g\notin B} h_g g = \sum_{g\in B} c_g \lambda_g g + \sum_{g\in B}(h_g - c_g\lambda_g)g + \sum_{g\notin B} h_g g, \tag{3.4}$$

where all terms appearing in the last two sums are $\prec u$.

From Lemma 3.2(i), we may suppose v_g is the term of g such that $u = lt(\lambda_g g) = \lambda_g v_g \succ \lambda_g v$ for any terms $v \neq v_g$ of g. Let d_g be the coefficient of g at v_g. Then by Lemma 3.1,

$$\begin{aligned}\sum_{g\in B} c_g\lambda_g g &= \sum_{g\in B} c_g\lambda_g d_g\left(\frac{g}{d_g}\right) = \sum_{g\in B} c_g(d'_g\lambda_g + \xi_g)\left(\frac{g}{d_g}\right)\\ &= \sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g}\right) + \sum_{g\in B} c_g\xi_g\left(\frac{g}{d_g}\right)\end{aligned} \tag{3.5}$$

for some elements $d'_g \in R$ and $\xi_g \in D$ with all terms appear in the last sum are $\prec u$. Also, by Lemma 3.1, all terms of ξ_g are in the same orthant as λ_g and $\prec' \lambda_g$. Then all terms appear in the last sum of (3.5) are $\prec' lt_{\prec'}(\lambda_g g) \preceq' lt_{\prec'}(f)$.

Note that u appears only in

$$\begin{aligned}\sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g}\right) &= \sum_{g\in B} c_g d'_g\lambda_g v_g + \sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g} - v_g\right)\\ &= \left(\sum_{g\in B} c_g d'_g\right)u + \sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g} - v_g\right)\end{aligned}$$

and all terms appearing in the last sum are $\prec u$. Since $lt(f) \prec u$ it follows that $\sum_{g\in B} c_g d'_g = 0$. Denote $\lambda_g(\frac{g}{d_g})$ by r_g, then

$$\sum_{g\in B} c_g d'_g\lambda_g\left(\frac{g}{d_g}\right) = \sum_{g\in B}(c_g d'_g)r_g = \sum_{i,k} b_{i,k}(r_{g_i} - r_{g_k}) \tag{3.6}$$

for some $g_i, g_k \in B$.

Since

$$r_{g_i} - r_{g_k} = \lambda_{g_i}\left(\frac{g_i}{d_{g_i}}\right) - \lambda_{g_k}\left(\frac{g_k}{d_{g_k}}\right)$$

and $\lambda_{g_i} v_{g_i} = \lambda_{g_k} v_{g_k} = u \in \Lambda_j E$ for an Λ_j, it follows from Lemma 3.3 that $v_{g_i} = lt_j(g_i)$, $v_{g_k} = lt_j(g_k)$, $d_{g_i} = lc_j(g_i)$, $d_{g_k} = lc_j(g_k)$, $\lambda_{g_i} = \frac{u}{lt_j(g_i)}$, $\lambda_{g_k} = \frac{u}{lt_j(g_k)}$ and then

$$r_{g_i} - r_{g_k} = \frac{u}{lt_j(g_i)}\frac{g_i}{lc_j(g_i)} - \frac{u}{lt_k(g_k)}\frac{g_k}{lc_j(g_k)}$$

with $lt(r_{g_i} - r_{g_k}) \prec u$.

Note that for all Λ_j, for all g_i, g_k and for all $v \in V(j, g_i, g_k)$, the S-polynomials $S(j, g_i, g_k, v)$ can be $\prec$-reduced to 0 modulo G relative to $\prec'$. We have

$$r_{g_i} - r_{g_k} = \sum_{g\in G} p_g g \tag{3.7}$$

with $lt(p_g g) \prec u$ and $lt_{\prec'}(p_g g) \preceq' \max_{\prec'}\{lt_{\prec'}(\lambda_{g_i} g_i), lt_{\prec'}(\lambda_{g_k} g_k)\} \preceq' lt_{\prec'}(f)$.

Replace the first sum on the right-hand side of (3.4) by (3.5), and replace the first sum in the right of (3.5) by (3.6); then replace $r_{g_i} - r_{g_k}$ on the right-hand side of (3.6) by (3.7). We get another form of $f = \sum_{g \in G} h'_g g$ such that

$$u \succ \max_{\prec}\{lt(h'_g g) \mid g \in G\} \quad \text{and} \quad lt_{\prec'}(h'_g g) \preceq' lt_{\prec'}(f),$$

which is a contradiction to the minimality of u. This completes the proof of the theorem. $\square$

Example 3.3. If W is a submodule of F generated by a finite set G and every $g \in G$ is a monomial, i.e. consists of only one term, then G is a relative Gröbner basis of W. In fact in this case all S-polynomials $S(j, g_i, g_k, v)$ and $S'(j, g_i, g_k, v)$ are 0. By Theorem 3.3 this implies that G is a relative Gröbner basis of W. $\square$

Following Theorem 3.3, the algorithm for computing a relative Gröbner basis can be divided into two parts. The first part deals with $S'(j, g_i, g_k, v)$ and determines a Gröbner basis w.r.t. $\prec'$. Then, the second part deals with $S(j, g_i, g_k, v)$ and determines a relative Gröbner basis. Similar to the algorithm for computing a Gröbner basis w.r.t. a generalized term order in Zhou and Winkler (2006) Theorem 3.3, we have the following algorithm.

Theorem 3.4 (*Buchberger's Algorithm for computing Relative Gröbner Bases*). *Let F be a free D-module, $\prec$ and $\prec'$ be two generalized term order on ΛE, G be a finite subset of $F \backslash \{0\}$ and W be the submodule in F generated by G. For each Λ_j and $f, g \in F \backslash \{0\}$ let $V(j, f, g)$, $S(j, f, g, v)$ and $S'(j, f, g, v)$ be as in Definition 3.4 w.r.t. $\prec$ and $\prec'$, respectively. Then by the following algorithm a $\prec$-Gröbner basis of W relative to $\prec'$ can be computed:*

Input: $G = \{g_1, \ldots, g_\mu\}$, *a set of generators of W*
$\prec$ *and* $\prec'$, *two generalized term orders on ΛE*
output: $G'' = \{g''_1, \ldots, g''_\nu\}$, *a $\prec$-Gröbner basis of W relative to $\prec'$*
Begin
$G' := G$ **;**
While *there exist $f, g \in G'$ and $v \in V(j, f, g)$ such that*
$S'(j, f, g, v)$ is reduced (w.r.t. $\prec'$) to $r \neq 0$ by G'
Do $G' := G' \cup \{r\}$
Endwhile ;
$G'' := G'$ **;**
While *there exist $f, g \in G''$ and $v \in V(j, f, g)$ such that*
$S(j, f, g, v)$ is $\prec$-reduced to $r \neq 0$ by G'' relative to $\prec'$
Do $G'' := G'' \cup \{r\}$
Endwhile
End $\square$

4. Computing difference-differential dimension polynomials using relative Gröbner bases

Let R be a Δ–Σ-field, D the ring of Δ–Σ-operators over R, M a finitely generated Δ–Σ-module (i.e. a finitely generated difference-differential module), F a finitely generated free Δ–Σ module. We will continue to use the notations and conventions of the preceding sections.

Now we consider difference-differential dimension polynomials $\psi_A(t_1, t_2)$ in two variables t_1 and t_2 by the approach of relative difference-differential Gröbner bases.

Choose the canonical orthant decomposition on $\mathbb{Z}^n$ as in Example 2.1 and define the generalized term orders "$\prec$" and "$\prec'$" on ΛE of the terms of F as follows: for $\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n}$ we set

$$|\lambda|_1 := k_1 + \cdots + k_m \quad \text{and} \quad |\lambda|_2 := |l_1| + \cdots + |l_n|;$$

also for $\lambda e_i \in \Lambda E$ we set

$$|\lambda e_i|_1 := |\lambda|_1 \quad \text{and} \quad |\lambda e_i|_2 := |\lambda|_2.$$

We write $<_{lex}$ for the lexicographic order.

Now for $\lambda e_i = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} e_i$ and $\mu e_j = \delta_1^{r_1} \cdots \delta_m^{r_m} \sigma_1^{s_1} \cdots \sigma_n^{s_n} e_j$ we define

$$\begin{aligned} \lambda e_i \prec \mu e_j \quad :\Longleftrightarrow \quad & (|\lambda|_2, |\lambda|_1, e_i, k_1, \ldots, k_m, |l_1|, \ldots, |l_n|, l_1, \ldots, l_n) \\ & <_{lex} \\ & (|\mu|_2, |\mu|_1, e_j, r_1, \ldots, r_m, |s_1|, \ldots, |s_n|, s_1, \ldots, s_n), \end{aligned}$$

and

$$\begin{aligned} \lambda e_i \prec' \mu e_j \quad :\Longleftrightarrow \quad & (|\lambda|_1, |\lambda|_2, e_i, k_1, \ldots, k_m, |l_1|, \ldots, |l_n|, l_1, \ldots, l_n) \\ & <_{lex} \\ & (|\mu|_1, |\mu|_2, e_j, r_1, \ldots, r_m, |s_1|, \ldots, |s_n|, s_1, \ldots, s_n). \end{aligned}$$

For $u = \sum_{\lambda \in \Lambda} a_\lambda \lambda \in D$ we define

$$|u|_1 := \max\{|\lambda|_1 \mid a_\lambda \neq 0\} \quad \text{and} \quad |u|_2 := \max\{|\lambda|_2 \mid a_\lambda \neq 0\}.$$

We may consider D as a bifiltered ring with the bifiltration $(D_{rs})_{r,s \in \mathbb{Z}}$ such that $D_{rs} = \{u \in D \mid |u|_1 \leq r, \ |u|_2 \leq s\}$ for $r, s \in \mathbb{N}$ and $D_{rs} = \{\}$ if at least one of the numbers r, s is negative. Obviously $\bigcup\{D_{rs} \mid r, s \in \mathbb{Z}\} = D$, $D_{rs} \subseteq D_{r+1,s}$, $D_{rs} \subseteq D_{r,s+1}$ for any $r, s \in \mathbb{Z}$ and $D_{kl} D_{rs} = D_{r+k,s+l}$ for any $r, s, k, l \in \mathbb{Z}$.

Let M be a finitely generated left D-module with generators $h_1, \ldots, h_q$. Let

$$M_{rs} = D_{rs} h_1 + \cdots + D_{rs} h_q$$

for any $r, s \in \mathbb{Z}$. Then $(M_{rs})_{r,s \in \mathbb{Z}}$ is an excellent bifiltration of M, i.e. every (M_{rs}) is a finitely generated R-module and $D_{kl} M_{rs} = M_{r+k,s+l}$.

Definition 4.1. A polynomial $\psi(t_1, t_2)$ in $\mathbb{Q}[t_1, t_2]$ is called a *(bivariate) numerical* if $\psi(t_1, t_2) \in \mathbb{Z}$ for all sufficiently large $(r_1, r_2) \in \mathbb{Z}^2$, i.e. there exists a tuple $(s_1, s_2) \in \mathbb{Z}^2$ such that $\psi(r_1, r_2) \in \mathbb{Z}$ for all integers $r_1, r_2 \in \mathbb{Z}$ with $r_i \geq s_i$ $(1 \leq i \leq 2)$.

The numerical polynomial $\psi(t_1, t_2)$ is called *(bivariate) difference-differential dimension polynomial* associated with M, if

(i) $deg\ \psi \leq m + n$, $deg_{t_1} \psi \leq m$, and $deg_{t_2} \psi \leq n$ and
(ii) $\psi(t_1, t_2) = dim_R M_{t_1,t_2}$ for all sufficiently large $t_1, t_2 \in \mathbb{N}$. $\square$

Levin (2000) investigated bivariate difference-differential dimension polynomials using the characteristic set. The method of Levin is rather delicate but no algorithm for computing the characteristic set is described. We will show that, by the method of relative difference-differential Gröbner bases, the bivariate difference-differential dimension polynomials can be computed.

Theorem 4.1. *Let R be a Δ–Σ-field, D and M be as above, in particular let M have the generators $h_1, \ldots, h_q$. Let F be a free Δ–Σ module with a basis $e_1, \ldots, e_q$ and $\pi : F \longrightarrow M$ the natural Δ–Σ epimorphism of F onto M ($\pi(e_i) = h_i$ for $i = 1, \ldots, q$).*

Let $\prec$ and $\prec'$ be the generalized term orders on ΛE of the terms of F defined above. Consider the submodule $N = \ker \pi$ of F and let $G = \{g_1, \ldots, g_p\}$ be a $\prec$-Gröbner basis of N relative to $\prec'$. Let

$$\begin{aligned} U_{r,s} = &\left\{ w \in \Lambda E \;\middle|\; |w|_1 \le r, |w|_2 \le s, \text{ and } w \ne lt_\prec(\lambda g_i) \text{ for all } \lambda \in \Lambda, g_i \in G \right\} \\ &\cup \\ &\left\{ w \in \Lambda E \;\middle|\; |w|_1 \le r, |w|_2 \le s, \right. \\ &\qquad\qquad \left. \text{and } |lt_{\prec'}(\lambda g_i)|_1 > r \text{ for all } \lambda \in \Lambda, g_i \in G \text{ s.t. } w = lt_\prec(\lambda g_i) \right\}. \end{aligned}$$

Then the bivariate difference-differential dimension polynomial ψ associated with M is the cardinality of U, i.e.

$$\psi(r, s) = \left| U_{r,s} \right|.$$

Proof. First, let us show that every element λh_i ($i = 1, \ldots, q$, $\lambda \in \Lambda$, $|\lambda|_1 \le r$, $|\lambda|_2 \le s$), that does not belong to $\pi(U_{r,s})$, can be written as a finite linear combination of elements of $\pi(U_{r,s})$ with coefficients from R. $\lambda h_i \notin \pi(U_{r,s})$ implies $\lambda e_i \notin U_{r,s}$, so we have $\lambda e_i = lt_\prec(\lambda' g_j)$ for some $\lambda' \in \Lambda$, $g_j \in G$, and $|[lt_{\prec'}(\lambda' g_j)]|_1 \le r$. Therefore

$$\lambda' g_j = a_j \lambda e_i + \sum_\nu a_\nu \lambda_\nu e_\nu,$$

where $a_j \ne 0$ and $a_\nu \ne 0$ for finitely many a_ν. Obviously, $\lambda_\nu e_\nu \prec \lambda e_i = lt_\prec(\lambda' g_j)$. Then by the definition of $\prec$, $|\lambda_\nu|_2 \le s$. On the other hand, since $|[lt_{\prec'}(\lambda' g_j)]|_1 \le r$ and $\lambda_\nu e_\nu \prec' lt_{\prec'}(\lambda' g_j)$, it follows from the definition of $\prec'$ that $|\lambda_\nu|_1 \le r$. Now note that $G \subseteq N = \ker(\pi)$, we have $0 = \pi(g_j)$ and

$$0 = \lambda' \pi(g_j) = \pi(\lambda' g_j) = a_j \pi(\lambda e_i) + \sum_\nu a_\nu \pi(\lambda_\nu e_\nu) = a_j \lambda h_i + \sum_\nu a_\nu \lambda_\nu h_\nu.$$

So we see that λh_i is a finite linear combination with coefficients from R of some elements of the form $\lambda_\nu h_\nu$ ($1 \le \nu \le q$) such that $|\lambda_\nu|_1 \le r$, $|\lambda_\nu|_2 \le s$ and $\lambda_\nu e_\nu \prec \lambda e_i$.

If there are some $\lambda_\nu h_\nu \notin \pi(U_{r,s})$, then we may repeat the same procedure with λh_i replaced by $\lambda_\nu h_\nu$. Thus, by induction on λe_j ($\lambda \in \Lambda$, $1 \le \nu \le q$) with respect to the order $\prec$ we obtain that

$$\lambda h_i = \sum_\mu b_\mu \lambda_\mu h_\mu$$

such that $|\lambda_\mu|_1 \le r$, $|\lambda_\mu|_2 \le s$ and $\lambda_\mu h_\mu \in \pi(U_{r,s})$ for all μ.

Now we have to prove that the set $\pi(U_{r,s})$ is linearly independent over R. Suppose that $\sum_{i=1}^l a_i \pi(u_i) = 0$ for some $u_1, \ldots, u_l \in U_{r,s}$, $a_1, \ldots, a_l \in R$. Then $h = \sum_{i=1}^l a_i u_i \in N$. By the definition of $U_{r,s}$ we see that

$$u_i \notin \left\{ \bigcup_{i=1,\ldots,p} \left\{ lt_\prec(\lambda g_i) \;\middle|\; |lt_{\prec'}(\lambda g_i)|_1 \le r, \lambda \in \Lambda \right\} \right\}.$$

This means

$$lt_{\prec}(h) \notin \left\{ \bigcup_{i=1,\ldots,p} \left\{ lt_{\prec}(\lambda g_i) \mid lt_{\prec'}(\lambda g_i) \preceq' lt_{\prec'}(h), \lambda \in \Lambda \right\} \right\}.$$

In fact, $|lt_{\prec'}(h)|_1 = |u_j|_1 \le r$ and if $u_i = lt_{\prec}(h) = lt_{\prec}(\lambda g_i)$ then $|lt_{\prec'}(\lambda g_i)|_1 > r$. So by the definition of $\prec'$ we have $lt_{\prec'}(\lambda g_i) \not\preceq' lt_{\prec'}(h)$.

Therefore, h is $\prec$-reduced modulo G relative to $\prec'$. By Proposition 3.1(iii) we get $h = 0$ and $a_i = 0, i = 1, \ldots, l$. So $\pi(U_{r,s})$ is linearly independent over R. Actually π is a bijection from $U_{r,s}$ to $\pi(U_{r,s})$. So

$$\psi(r, s) = dim_R M_{r,s} = \left|\pi(U_{r,s})\right| = \left|U_{r,s}\right|.$$

This completes the proof of the theorem. □

The difference-differential dimension polynomial $\psi(t_1, t_2)$ carries more invariants than the "one variable" dimension polynomial $\phi(t)$. From the point of view of strength of systems of difference-differential equations, the polynomial $\psi(t_1, t_2)$ determines the strength of systems w.r.t. each of the sets of operators Δ and Σ while the polynomial $\phi(t)$ determines just the general strength of the systems w.r.t. the set $\Delta \bigcup \Sigma$.

Example 4.1. Let R be a difference-differential field whose basic sets Δ and Σ consist of a single δ and a single σ. Furthermore, let D be the ring of Δ–Σ-operators over R and $M = Dh$ be a Δ–Σ module whose generator h satisfies the defining equation

$$(\delta\sigma + \sigma^{-2})h = 0.$$

In other words, M is isomorphic to the factor module of a free Δ–Σ module F with a free generator e by its Δ–Σ submodule N which is a cyclic submodule with a generator $\{g = \delta\sigma + \sigma^{-2}\}$. We compute the difference-differential dimension polynomial $\psi(r, s)$. By Theorem 4.1, we need to compute a relative Gröbner basis of N and then $\psi(r, s) = \left|U_{r,s}\right|$.

Clearly the relative Gröbner basis is $\{g = \delta\sigma + \sigma^{-2}\}$ (compare Example 3.2). We have $lt(g) = \sigma^{-2} \in \Lambda_2$. As the leading term of $\sigma g = \delta\sigma^2 + \sigma^{-1}$ is $\delta\sigma^2 \in \Lambda_1$, we have

$$lt(\lambda g) = \Lambda_1 \delta\sigma^2 \bigcup \Lambda_2 \sigma^{-2}.$$

Put

$$U'_{r,s} := \left\{ w \in \Lambda \mid |w|_1 \le r, |w|_2 \le s, \text{ and } w \ne lt_{\prec}(\lambda g) \text{ for all } \lambda \in \Lambda \right\},$$

$$U''_{r,s} := \left\{ w \in \Lambda \mid |w|_1 \le r, |w|_2 \le s, \text{ and } |lt_{\prec'}(\lambda g)|_1 > r \text{ for all } \lambda \in \Lambda \text{ s.t. } w = lt_{\prec}(\lambda g) \right\}.$$

Then

$$\left|U_{r,s}\right| = \left|U'_{r,s}\right| + \left|U''_{r,s}\right|$$

and

$$\psi(r, s) = dim_R M_{r,s} = \left|U_{r,s}\right| = (3r + s + 2) + (s - 1) = 3r + 2s + 1. \quad \square$$

Example 4.2. Let R be a difference-differential field whose basic sets $\Delta = \{\delta_1, \delta_2\}$ and $\Sigma = \{\sigma\}$. Let D be the ring of Δ–Σ-operators over R and $M = Dh$ be a Δ–Σ module whose generator h satisfies the following defining equations

$$(\delta_1^4\delta_2\sigma^{-3} + \delta_1^2\delta_2\sigma^3)h = 0,$$
$$(\delta_1^2\delta_2\sigma^2 - \delta_1^2\delta_2\sigma^{-4})h = 0.$$

Then M is isomorphic to the factor module of a free Δ–Σ module F with a free generator e by the Δ–Σ submodule N generated by

$$\{g_1 = \delta_1^4\delta_2\sigma^{-3} + \delta_1^2\delta_2\sigma^3,\ g_2 = \delta_1^2\delta_2\sigma^2 - \delta_1^2\delta_2\sigma^{-4}\}.$$

Now the generalized term orders "$\prec$", "$\prec'$" are defined as:

$$\delta_1^{k_1}\delta_2^{k_2}\sigma^l \prec \delta_1^{r_1}\delta_2^{r_2}\sigma^s \iff (|l|, k_1+k_2, k_1, k_2, l) <_{lex} (|s|, r_1+r_2, r_1, r_2, s),$$
$$\delta_1^{k_1}\delta_2^{k_2}\sigma^l \prec' \delta_1^{r_1}\delta_2^{r_2}\sigma^s \iff (k_1+k_2, |l|, k_1, k_2, l) <_{lex} (r_1+r_2, |s|, r_1, r_2, s).$$

For computing the difference-differential dimension polynomial $\psi(t_1, t_2)$ we first computing a relative Gröbner basis of N. Let $\Lambda_1 = \{\delta_1^{k_1}\delta_2^{k_2}\sigma^l \mid l \geq 0\}$ and $\Lambda_2 = \{\delta_1^{k_1}\delta_2^{k_2}\sigma^l \mid l \leq 0\}$.

Since

$$\sigma^3 g_1 = \delta_1^4\delta_2 + \delta_1^2\delta_2\sigma^6,$$
$$\sigma g_2 = \delta_1^2\delta_2\sigma^3 - \delta_1^2\delta_2\sigma^{-3} \quad \text{and} \quad g_2 = \delta_1^2\delta_2\sigma^2 - \delta_1^2\delta_2\sigma^{-4},$$

it follows that

$$lt_{\prec'}(\sigma^3 g_1) = \delta_1^4\delta_2 \in \Lambda_1 \bigcap \Lambda_2,$$
$$lt_{\prec'}(\sigma g_2) = \delta_1^2\delta_2\sigma^3 \in \Lambda_1 \quad \text{and} \quad lt_{\prec'}(g_2) = \delta_1^2\delta_2\sigma^{-4} \in \Lambda_2.$$

Then we may see that

$$\{\lambda \in \Lambda \mid lt_{\prec'}(\lambda g_1) \in \Lambda_1\} = \Lambda_1\sigma^3 \qquad \{\eta \in \Lambda \mid lt_{\prec'}(\eta g_2) \in \Lambda_1\} = \Lambda_1\sigma$$

and

$$\{lt_{\prec'}(\lambda g_1) \in \Lambda_1 \mid \lambda \in \Lambda\} = \Lambda_1\delta_1^4\delta_2 \qquad \{lt_{\prec'}(\eta g_2) \in \Lambda_1 \mid \eta \in \Lambda\} = \Lambda_1\delta_1^2\delta_2\sigma^3.$$

Therefore $V'(1, g_1, g_2) = \{v_1'\} = \{\delta_1^4\delta_2\sigma^3\}$ and by Definition 3.4,

$$S'(1, g_1, g_2, v_1') = \sigma^6 g_1 - \delta_1^2\sigma g_2 = \delta_1^2\delta_2\sigma^9 + \delta_1^4\delta_2\sigma^{-3}.$$

Since $lt_{\prec'}(\delta_1^2\delta_2\sigma^9 + \delta_1^4\delta_2\sigma^{-3}) = \delta_1^4\delta_2\sigma^{-3} = lt_{\prec'}(g_1)$, $S'(1, g_1, g_2, v_1')$ can be reduced to $\delta_1^2\delta_2\sigma^9 - \delta_1^2\delta_2\sigma^3$ mod g_1, and then it can be reduced to 0 mod g_2 since $\delta_1^2\delta_2\sigma^9 - \delta_1^2\delta_2\sigma^3 = \sigma^7 g_2$.

Similarly we have

$$\{\lambda \in \Lambda \mid lt_{\prec'}(\lambda g_1) \in \Lambda_2\} = \Lambda_2\sigma^3 \qquad \{\eta \in \Lambda \mid lt_{\prec'}(\eta g_2) \in \Lambda_2\} = \Lambda_2$$
$$\{lt_{\prec'}(\lambda g_1) \in \Lambda_2 \mid \lambda \in \Lambda\} = \Lambda_2\delta_1^4\delta_2 \qquad \{lt_{\prec'}(\eta g_2) \in \Lambda_2 \mid \eta \in \Lambda\} = \Lambda_2\delta_1^2\delta_2\sigma^{-4}$$
$$V'(2, g_1, g_2) = \{v_2'\} = \{\delta_1^4\delta_2\sigma^{-4}\}.$$

Then

$$\begin{aligned} S'(2, g_1, g_2, v_2') &= \sigma^{-1} g_1 + \delta_1^2 g_2 = \delta_1^2\delta_2\sigma^2 + \delta_1^4\delta_2\sigma^2 \\ &= \sigma^5 g_1 + \delta_1^2\delta_2\sigma^2 - \delta_1^2\delta_2\sigma^8 = \sigma^5 g_1 - \sigma^6 g_2 \end{aligned}$$

which is reduced to 0 by $\{g_1, g_2\}$. By Theorem 3.2, $\{g_1, g_2\}$ is a Gröbner basis with respect to $\prec'$ of N.

Now we compute S-polynomials with respect to $\prec$.

$$\begin{aligned}
&\sigma g_1 = \delta_1^4\delta_2\sigma^{-2} + \delta_1^2\delta_2\sigma^4 \quad \text{and} \quad g_1 = \delta_1^4\delta_2\sigma^{-3} + \delta_1^2\delta_2\sigma^3,\\
&lt_\prec(\sigma_1^g) = \delta_1^2\delta_2\sigma^4 \in \Lambda_1 \quad \text{and} \quad lt_\prec(g_1) = \delta_1^4\delta_2\sigma^{-3} \in \Lambda_2\\
&\sigma g_2 = \delta_1^2\delta_2\sigma^3 - \delta_1^2\delta_2\sigma^{-3} \quad \text{and} \quad g_2 = \delta_1^2\delta_2\sigma^2 - \delta_1^2\delta_2\sigma^{-4},\\
&lt_\prec(\sigma g_2) = \delta_1^2\delta_2\sigma^3 \in \Lambda_1 \quad \text{and} \quad lt_\prec(g_2) = \delta_1^2\delta_2\sigma^{-4} \in \Lambda_2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\{\lambda \in \Lambda \mid lt_\prec(\lambda g_1) \in \Lambda_1\} = \Lambda_1\sigma \quad \{\eta \in \Lambda \mid lt_\prec(\eta g_2) \in \Lambda_1\} = \Lambda_1\sigma\\
&\{lt_\prec(\lambda g_1) \in \Lambda_1 \mid \lambda \in \Lambda\} = \Lambda_1\delta_1^2\delta_2\sigma^4 \quad \{lt_\prec(\eta g_2) \in \Lambda_1 \mid \eta \in \Lambda\} = \Lambda_1\delta_1^2\delta_2\sigma^3
\end{aligned}$$

and

$$\begin{aligned}
&\{\lambda \in \Lambda \mid lt_\prec(\lambda g_1) \in \Lambda_2\} = \Lambda_2 \quad \{\eta \in \Lambda \mid lt_\prec(\eta g_2) \in \Lambda_2\} = \Lambda_2\\
&\{lt_\prec(\lambda g_1) \in \Lambda_2 \mid \lambda \in \Lambda\} = \Lambda_2\delta_1^4\delta_2\sigma^{-3} \quad \{lt_\prec(\eta g_2) \in \Lambda_2 \mid \eta \in \Lambda\} = \Lambda_2\delta_1^2\delta_2\sigma^{-4}.
\end{aligned}$$

Then

$$\begin{aligned}
&V(1, g_1, g_2) = \{v_1\} = \{\delta_1^2\delta_2\sigma^4\}\\
&V(2, g_1, g_2) = \{v_2\} = \{\delta_1^4\delta_2\sigma^{-4}\}.
\end{aligned}$$

By Definition 3.4 we have

$$\begin{aligned}
&S(1, g_1, g_2, v_1) = \sigma g_1 - \sigma^2 g_2 = \delta_1^4\delta_2\sigma^{-2} + \delta_1^2\delta_2\sigma^{-2}.\\
&S(2, g_1, g_2, v_2) = \sigma^{-1}g_1 + \delta_1^2 g_2 = \delta_1^4\delta_2\sigma^2 + \delta_1^2\delta_2\sigma^2.
\end{aligned}$$

Since $lt_\prec(S(1, g_1, g_2, v_1)) = \delta_1^4\delta_2\sigma^{-2} \notin \{\Lambda_2\delta_1^4\delta_2\sigma^{-3}\}\bigcup\{\Lambda_2\delta_1^2\delta_2\sigma^{-4}\}$, we see that $S(1, g_1, g_2, v_1)$ is reduced w.r.t. $\{g_1, g_2\}$. Denote it by g_3, then $S(2, g_1, g_2, v_2) = \sigma^4 g_3$ can be reduced relatively mod g_3 to 0.

Put $G = \{g_1, g_2, g_3\}$. In a similar way we get the S-polynomials of G as follows:

$$\begin{aligned}
&S(1, g_1, g_3, v_3) = \delta_1^2\sigma g_1 - \sigma^6 g_3 = \delta_1^6\delta_2\sigma^{-2} - \delta_1^2\delta_2\sigma^4\\
&S(2, g_1, g_3, v_4) = g_1 - \sigma^{-1}g_3 = \delta_1^2\delta_2\sigma^3 - \delta_1^2\delta_2\sigma^{-3}\\
&S(1, g_2, g_3, v_5) = \delta_1^2\sigma g_2 - \sigma^5 g_3 = -\delta_1^4\delta_2\sigma^{-3} - \delta_1^2\delta_2\sigma^3\\
&S(2, g_2, g_3, v_6) = \delta_1^2 g_2 + \sigma^{-2}g_3 = \delta_1^4\delta_2\sigma^2 + \delta_1^2\delta_2\sigma^{-4}.
\end{aligned}$$

Since $lt_\prec(S(1, g_1, g_3, v_3)) = \delta_1^2\delta_2\sigma^4 = lt_\prec(\sigma g_1)$, and $lt_{\prec'}(\sigma g_1) = \delta_1^4\delta_2\sigma^{-2} \prec' \delta_1^6\delta_2\sigma^{-2} = lt_{\prec'}(S(1, g_1, g_3, v_3))$, we see that $S(1, g_1, g_2, v_3)$ can be $\prec$-reduced to $\delta_1^6\delta_2\sigma^{-2} + \delta_1^4\delta_2\sigma^{-2}$ modulo g_1 relative to $\prec'$. Then it can be reduced relatively modulo g_3 to 0, since $\delta_1^6\delta_2\sigma^{-2} + \delta_1^4\delta_2\sigma^{-2} = \delta_1^2 g_3$.

Similarly $S(2, g_2, g_3, v_6) = -g_2 + \sigma^4 g_3$ can be relatively reduced to 0 modulo G. Also $S(2, g_1, g_3, v_4) = \sigma g_2$, $S(1, g_2, g_3, v_5) = -g_1$. So these S-polynomials can be relatively reduced to 0 modulo G. Therefore, by Theorem 3.4, G is a relative Gröbner basis of N.

It follows from Theorem 4.1 that the difference-differential dimension polynomial $\psi(r, s)$ is determined by

$$\psi(r, s) = |U_{r,s}| = |U'_{r,s}| + |U''_{r,s}|,$$

where

$$U'_{r,s} := \left\{ w \in \Lambda \,\middle|\, |w|_1 \le r, |w|_2 \le s, \text{ and } w \ne lt_{\prec}(\lambda g_i) \text{ for all } \lambda \in \Lambda, g_i \in G \right\},$$

$$U''_{r,s} := \Big\{ w \in \Lambda \,\Big|\, |w|_1 \le r, |w|_2 \le s, \\ \text{and } |lt_{\prec'}(\lambda g_i)|_1 > r \text{ for all } \lambda \in \Lambda, g_i \in G \text{ s.t. } w = lt_{\prec}(\lambda g_i) \Big\}.$$

From the fact $lt_{\prec}(\lambda g_2) = lt_{\prec'}(\lambda g_2)$, $lt_{\prec}(\lambda g_3) = lt_{\prec'}(\lambda g_3)$ we see that for $j = 2, 3$ there is no $w \in \Lambda$ such that $|w|_1 \le r$, $w = lt_{\prec}(\lambda g_j)$ and $lt_{\prec'}(\lambda g_j)]|_1 > r$. Furthermore, if $\lambda \in \Lambda_2$ then $lt_{\prec}(\lambda g_1) = lt_{\prec'}(\lambda g_1)$; if $\lambda \in \Lambda_1$ then $\{w \in \Lambda | w = lt_{\prec}(\lambda \sigma g_1)\} \subset \{w \in \Lambda | w = lt_{\prec}(\lambda \sigma g_2)\}$. This means that the condition in $U''_{r,s}$ does not hold for such w. So we conclude that $U''_{r,s} = \emptyset$. Thus, finally, from

$$\begin{aligned} U'_{r,s} = \{w = \delta_1^{k_1}\delta_2^{k_2}\sigma^l \mid\ & k_1 + k_2 \le r,\ |\, l \,| \le s, \\ & (k_1, k_2, l) \notin (2, 1, 3) + \{\mathbb{N}^3\},\ (k_1, k_2, l) \notin (2, 1, -4) + \{\mathbb{N}^2 \times (-\mathbb{N})\} \\ & (k_1, k_2, l) \notin (4, 1, 0) + \{\mathbb{N}^2 \times \mathbb{Z}\}\} \end{aligned}$$

we get for all sufficiently large $r, s \in \mathbb{N}$

$$\begin{aligned} \psi(r, s) &= \left|U'_{r,s}\right| \\ &= (r + 1)(2s + 1) + (r - 3)[2(2s + 1) + 12] + [2(2s + 1) + 6] \\ &\quad + [2(2s + 1)] + (2s + 1) \\ &= 6rs + 15r - 30. \quad \square \end{aligned}$$

Example 4.3. Let R , Δ, Σ and D be the same as in Example 4.1. Let $M = Dh_1 + Dh_2$ be a Δ–Σ module whose generators h_1, h_2 satisfy the defining equations

$$\delta\sigma h_1 + \sigma^{-2}h_2 = 0,$$
$$\delta^2\sigma h_1 + \delta h_2 = 0.$$

Then M is isomorphic to the factor module of a free Δ–Σ module F with free generators e_1, e_2 by its Δ–Σ submodule N generated by

$$\{g_1 = \delta\sigma e_1 + \sigma^{-2}e_2,\ g_2 = \delta^2\sigma e_1 + \delta e_2\}.$$

We compute the relative Gröbner basis of N , the cardinality of $U_{r,s}$ and $\psi(r, s)$.

Similarly as in Example 4.2, we get

$$S'(1, g_1, g_2, v_1) = S'(2, g_1, g_2, v_2) = \delta g_1 - g_2 = \delta\sigma^{-2}e_2 - \delta e_2 = g_3.$$

Since $lt_{\prec'}(\lambda g_1) \in \Lambda e_1$, $lt_{\prec'}(\lambda g_2) \in \Lambda e_1$ and $lt_{\prec'}(\lambda g_3) \in \Lambda e_2$, we see that $S'(k, g_i, g_3, v_s) = 0$, for all $i = 1, 2, k = 1, 2$. So $G' = \{g_1, g_2, g_3\}$ is a Gröbner basis with respect to $\prec'$ of N.

We compute S-polynomials with respect to $\prec$ as follows:

$$\begin{aligned} \sigma g_1 &= \underline{\delta\sigma^2 e_1} + \sigma^{-1}e_2, & g_1 &= \delta\sigma e_1 + \underline{\sigma^{-2}e_2}, \\ g_2 &= \underline{\delta^2\sigma e_1} + \delta e_2, & \sigma^{-1}g_2 &= \delta^2 e_1 + \underline{\delta\sigma^{-1}e_2}, \\ \sigma g_3 &= \delta\sigma^{-1}e_2 - \underline{\delta\sigma e_2}, & g_3 &= \underline{\delta\sigma^{-2}e_2} - \delta e_2 \end{aligned}$$

(underlined terms denote leading terms). Then

$$S(1, g_1, g_2, v_{12}^{(1)}) = \delta\sigma g_1 - \sigma g_2 = \delta\sigma^{-1}e_2 - \delta\sigma e_2 = \sigma g_3$$
$$S(2, g_1, g_2, v_{12}^{(2)}) = \delta g_1 - \sigma^{-2}g_2 = \delta^2\sigma e_1 - \delta^2\sigma^{-1}e_1$$

which can be reduced relatively mod g_2 to $\delta^2\sigma^{-1}e_1 + \delta e_2 = g_4$. Then

$$\sigma g_4 = \delta^2 e_1 + \underline{\delta\sigma e_2} \quad g_4 = \underline{\delta^2\sigma^{-1}e_1} + \delta e_2$$

and

$$\begin{aligned}
&S(1, g_1, g_3, v_{13}^{(1)}) = 0, \qquad S(1, g_2, g_3, v_{23}^{(1)}) = 0 \\
&S(2, g_1, g_3, v_{13}^{(2)}) = \delta g_1 - g_3 = \delta^2\sigma e_1 + \delta e_2 = g_2 \\
&S(2, g_2, g_3, v_{23}^{(2)}) = \sigma^{-2} g_1 - g_3 = \delta^2\sigma^{-1} e_1 + \delta e_2 = g_4 \\
&S(1, g_1, g_4, v_{14}^{(1)}) = 0, \qquad S(2, g_1, g_4, v_{14}^{(2)}) = 0 \\
&S(1, g_2, g_4, v_{24}^{(1)}) = 0, \qquad S(2, g_2, g_4, v_{24}^{(2)}) = 0 \\
&S(2, g_3, g_4, v_{34}^{(2)}) = 0 \\
&S(1, g_3, g_4, v_{34}^{(1)}) = \sigma g_3 + \sigma g_4 = \delta^2 e_1 + \delta\sigma^{-1} e_2 = \sigma^{-1} g_2.
\end{aligned}$$

So $G = \{g_1, g_2, g_3, g_4\}$ is a $\prec$-Gröbner basis of N relative to $\prec'$.

Now we determine the dimension polynomial

$$\psi(r, s) = \left|U_{r,s}\right| = \left|U'_{r,s}\right| + \left|U''_{r,s}\right|.$$

From

$$\begin{aligned}
U'_{r,s} &= \left\{w \in \Lambda E \;\middle|\; |w|_1 \le r, |w|_2 \le s \text{ and } w \ne lt_\prec(\lambda g_i) \text{ for all } \lambda \in \Lambda, g_i \in G\right\} \\
&= \left\{w \in \Lambda e_1 \;\middle|\; |w|_1 \le r, |w|_2 \le s \right. \\
&\qquad\qquad \left. \text{and } w \ne lt_\prec(\lambda\sigma g_1), w \ne lt_\prec(\lambda g_2), w \ne lt_\prec(\lambda\sigma g_4)\right\} \\
&\quad \cup \\
&\quad \left\{w \in \Lambda e_2 \;\middle|\; |w|_1 \le r, |w|_2 \le s \right. \\
&\qquad\qquad \text{and } w \ne lt_\prec(\lambda g_1), w \ne lt_\prec(\lambda\sigma^{-1} g_2), w \ne lt_\prec(\lambda\sigma g_3), \\
&\qquad\qquad \left. w \ne lt_\prec(\lambda g_3), w \ne lt_\prec(\lambda\sigma g_4)\right\}
\end{aligned}$$

we see

$$\left|U'_{r,s}\right| = (2s + 1) + (s + 2) + (r - 1) + (s + 2) + r = 2r + 4s + 4.$$

For determining $\left|U''_{r,s}\right|$ we only need to consider $g_1, \sigma^{-1}g_2, \sigma g_4$, because $lt_\prec(\lambda\sigma g_j) = lt_{\prec'}(\lambda\sigma g_j)$ for $j = 1, 3$, and $lt_\prec(\lambda g_j) = lt_{\prec'}(\lambda g_j)$ for $j = 2, 3, 4$.

$$\begin{aligned}
&\left\{w = lt_\prec(\lambda g_1) \;\middle|\; \lambda \in \Lambda_2 e_2, |w|_1 \le r, |w|_2 \le s, |lt_{\prec'}(\lambda g_1)|_1 > r\right\} \\
&\quad = \left\{w = \delta^k\sigma^{-l}\sigma^{-2}e_2 \;\middle|\; \le r, k + 1 > r\right\} \\
&\quad = \left\{w = \delta^r\sigma^{-(l+2)}e_2 \;\middle|\; l \ge 0\right\}.
\end{aligned}$$

But $w = \delta^r\sigma^{-(l+2)}e_2 = lt_\prec(\delta^{r-1}\sigma^{-l}g_3)$ and $|lt_{\prec'}(\delta^{r-1}\sigma^{-l}g_3)|_1 \le r$, so

$$\begin{aligned}
&\left\{w = lt_\prec(\lambda g_1) \;\middle|\; \lambda \in \Lambda_2 e_2, |w|_1 \le r, |w|_2 \le s, \right. \\
&\qquad \left. \text{and } |lt_{\prec'}(\lambda g_j)|_1 > r \text{ for all } \lambda \in \Lambda, g_j \in G \text{ s.t. } w = lt_\prec(\lambda g_j)\right\} \\
&\quad = \emptyset.
\end{aligned}$$

Similarly we get

$$\begin{aligned}&\big\{w = lt_{\prec}(\lambda\sigma^{-1}g_2) \;\big|\; \lambda \in \Lambda_2 e_2, |w|_1 \le r, |w|_2 \le s,\\ &\qquad\quad \text{and } |lt_{\prec'}(\lambda g_j)|_1 > r \text{ for all } \lambda \in \Lambda, g_j \in G \text{ s.t. } w = lt_{\prec}(\lambda g_j)\big\}\\ &\quad = \big\{\delta^r\sigma^{-(l+1)}e_2 \;\big|\; l = 0\big\},\end{aligned}$$

and finally

$$\begin{aligned}&\big\{w = lt_{\prec}(\lambda\sigma g_4) \;\big|\; \lambda \in \Lambda_1 e_2, |w|_1 \le r, |w|_2 \le s,\\ &\qquad\quad \text{and } |lt_{\prec'}(\lambda g_j)|_1 > r \text{ for all } \lambda \in \Lambda, g_j \in G \text{ s.t. } w = lt_{\prec}(\lambda g_j)\big\}\\ &\quad = \big\{\delta^r\sigma^{l+1}e_2 \;\big|\; l \ge 0, l+1 \le s\big\}.\end{aligned}$$

Combining all these partial results, we see that

$$\left|U''_{r,s}\right| = 0 + 1 + s = s + 1,$$

and therefore

$$\psi(r,s) = \left|U_{r,s}\right| = \left|U'_{r,s}\right| + \left|U''_{r,s}\right| = 2r + 5s + 5$$

for all sufficiently large $r, s \in \mathbb{N}$. $\square$

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